Motion Generated by an Oscillating Plate Contacting a Bingham Body

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A problem of some interest is the prediction of the influence of fluid rheology upon the propagation of oscillations in a fluid. Such predictions have formed the basis of useful turbulence models in Newtonian fluids (Gill and Scher, 1961; Hanks, 1968; Van Driest, 1956). As a beginning effort in this direction for non-Newtonian fluids, the calculation of the motion generated by a flat plate oscillating in its own plane will be considered for the case where it contacts fluid whose rheology is described by the Bingham plastic model:

$$\tau_{xy} = \tau_0 - \eta \frac{dv_x}{dy}, \quad \tau_{xy} > \tau_0 \tag{1}$$

$$0 = dv_x/dy, \quad \tau_{xy} \leqq \tau_0$$

The equation of motion pertinent to this simple one-dimensional problem is

$$\rho \frac{\partial v_x}{\partial t} = \eta \frac{\partial^2 v_x}{\partial y^2} \tag{2}$$

which may be expressed in terms of the dimensionless variables $\theta = tv_0^2/\nu$, $u = v_x/v_0$, $\xi = yv_0/\nu$, as

$$\frac{\partial u}{\partial \theta} = \frac{\partial^2 u}{\partial \xi^2} \tag{3}$$

where $v_0 = \sqrt{\tau_0/\rho}$ and $\nu = \eta/\rho$. The pertinent boundary conditions for this problem are $u = \exp[i(\omega\theta + \epsilon)]$ at $\xi = 0$, and $\partial u/\partial \xi = 0$ for $\xi \geqq \beta = hv_0/\nu$, where ω is the oscillatory frequency of the plate.

A solution which satisfies Equation (3) is

$$u(\theta, \xi) = e^{i(\omega\theta + \epsilon)}\, \phi(\xi) \tag{4}$$

where $\phi(\xi)$ satisfies the equation

$$\frac{d^2\phi}{d\xi^2} - i\omega\phi = 0 \tag{5}$$

together with the boundary conditions $\phi(0) = 1$, $\phi'(\beta) = 0$. The solution of Equation (5) which satisfies these two boundary conditions is

$$\phi(\xi) = \frac{e^{-\alpha\xi}\cos\alpha\xi}{1 + H(\beta)}\,(1 + H(\beta)e^{2\alpha\xi}) \tag{6}$$

where

$$H(\beta) = \frac{e^{-2\alpha\beta}(\cos\alpha\beta + \sin\alpha\beta)}{\cos\alpha\beta - \sin\alpha\beta} \tag{7}$$

and $\alpha = \sqrt{\omega/2}$. Thus, $u(\theta, \xi)$ is

$$u(\theta, \xi) = \frac{e^{-\alpha\xi}\cos\alpha\xi\cos\omega\theta}{1 + H(\beta)}\,(1 + H(\beta)e^{2\alpha\xi}), \quad \xi < \beta \tag{8}$$

$$U(\theta, \xi) = \frac{e^{-\alpha\beta}\cos\alpha\beta\cos\omega\theta\,(1 + H(\beta)e^{2\alpha\beta})}{1 + H(\beta)}, \quad \xi \geqq \beta \tag{9}$$

The shear stress exerted on the fluid by the plate is given by the expression

$$T(\theta) = 1 + \frac{\alpha\cos\omega\theta[1 - H(\beta)]}{1 + H(\beta)} \tag{10}$$

where $T(\theta) = \tau_w(\theta)/\tau_0$ and $\tau_w(\theta)$ is the instantaneous shear stress. The mean square value $\overline{T^2}$ taken over a time large compared with ω^{-1} is

$$\overline{T^2} = 1/2^{\alpha^2}\frac{[1 - H(\beta)]^2}{[1 + H(\beta)]^2} \tag{11}$$

Consideration of Equations (8) and (9) reveals the effect of the fluid rheology on this system. As β becomes large, Equation (8) goes over to the Newtonian solution given by Lamb (1945) in which the exponential damping causes the disturbance due to the oscillating plate to die out rapidly as one moves away from the plate. However, for moderate β Equation (8) more nearly resembles the case where a fixed plate near the oscillating one greatly increases the influence of the rheology on the wave (due to the retention of the positive exponential term). A significant difference exists, however, in the fact that the fluid rheology requires the boundary condition to be $\phi'(\beta) = 0$ rather than $\phi(\beta) = 0$ as in the fixed boundary problem of Lamb (1945). Equation (9) suggests that the unsheared bulk of the fluid would experience a solid body oscillation of frequency ω.

From these results it is to be expected that the rheology of this type of fluid will have a more pronounced influence on the behavior of oscillating flows than does the simple Newtonian fluid. This appears to be the case for turbulent pipe flows of non-Newtonian slurries (Hanks and Dadia, 1971).

LITERATURE CITED

Gill, W. N., and M. Scher, "A Modification of the Momentum Transport Hypothesis," *AIChE J.*, 7, 61 (1961).

Hanks, R. W., "On the Theoretical Calculation of Friction Factors for Laminar, Transitional, and Turbulent Flow of Newtonian Fluids in Pipes and Between Parallel Plane Walls," *ibid.*, 14, 691 (1968).

———., and B. H. Dadia, "Theoretical Analysis of the Turbulent Flow of Non-Newtonian Slurries in Pipes," *ibid.*, 17, 554 (1971).

Lamb, H., *Hydrodynamics*, 6th Edit., pp. 619-622, Dover (1945).

Van Driest, E. R., "On Turbulent Flow Near a Wall," *J. Aeronaut. Sci.*, p. 1007 (Nov., 1956).

Manuscript received September 28, 1973; revision received October 9 and accepted October 10, 1973.